IX Relativistische Edyn

$$\operatorname{div}\vec{B} = 0 \qquad \operatorname{rot}\vec{E} = -\frac{1}{c}\partial_t\vec{B}$$

$$\vec{B} = \operatorname{rot}\vec{A} \qquad \vec{E} = -\operatorname{grad}\phi - \frac{1}{c}\partial_t\vec{A}$$

$$\operatorname{div}\vec{E} = 4\pi\rho \qquad \operatorname{rot}\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\vec{E}$$

$$\longrightarrow \quad \Box\phi + \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\phi + \operatorname{div}\vec{A}\right) = -\frac{4\pi}{c}c\rho$$

$$\Box\vec{A} - \operatorname{grad}\left(\frac{1}{c}\partial_t\phi + \operatorname{div}\vec{A}\right) = -\frac{4\pi}{c}\vec{j}$$

Viererstrom $j^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} = \begin{pmatrix} \rho\, c \\ \rho\, \vec{u} \end{pmatrix} = \underbrace{\frac{\rho}{\gamma}}_{\rho_{ruhe}} \gamma \begin{pmatrix} c \\ \vec{u} \end{pmatrix} = \rho_{ruhe} \cdot u^\mu$

$\rho_{ruhe} \cdot \gamma = \rho \quad \longrightarrow \quad$ Zunahme d. Ladung auf verk. Strecke

mit $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \partial_i\right)$

& $A^\mu = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix}$

$/\!/ \; \Box = -\frac{1}{c^2}\partial_t^2 + \Delta$

$$\partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \partial_i\partial_i$$

$$\longrightarrow \Box = -\partial_\mu\partial^\mu = \partial_i\partial_i - \frac{1}{c^2}\partial_t^2$$

$\partial_\mu A^\mu = \frac{1}{c}\partial_t\phi + \partial_i A_i$

oberer Idx hat − in Ortsk.

$\partial^\nu\partial_\mu A^\mu = \begin{pmatrix} \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\phi + \partial_i A_i\right) \\ -\partial_i\left(\text{—} \; " \; \text{—} \right) \end{pmatrix}$

$$\longrightarrow -\partial_\mu\partial^\mu A^\nu + \partial^\nu\partial_\mu A^\mu = -\frac{4\pi}{c}j^\nu$$

Rausheben: $-\partial_\mu \underbrace{\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)}_{F^{\mu\nu}} = -\frac{4\pi}{c}j^\nu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = -F^{\nu\mu}$$

$$\longrightarrow \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

<u>homogenen MWG</u>

Levi-Civita : $\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} 1 & \text{gerade} \quad P \text{ v } 0123 \\ -1 & \text{unger.} \\ 0 & \text{sonst} \end{cases}$

012 2 vert ok

$\rightarrow 201$ bleibt

210 1 vert $-$

0123 $\quad 3120$ $\quad 3102$

3012 $\quad\quad\quad 3012$ $\quad$ 2 vert $-$ neg

versch bei 4er T macht es neg ?

$$\hat{F}^{30} = \underbrace{\tfrac{1}{2}\varepsilon^{3012}}_{\text{selbst - sein}} \underbrace{F_{12}}_{-B_z} + \underbrace{\tfrac{1}{2}\varepsilon^{3021}}_{\text{pos}} \underbrace{F_{21}}_{B_z}$$

$$= \frac{1}{2} B_z + \frac{1}{2} B_z = B_z \quad \checkmark$$

$$\longrightarrow \quad \hat{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

hom MWG $\quad \partial_\mu \hat{F}^{\mu\nu} = \emptyset$

$$\partial_\mu \frac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} \qquad F^{\lambda\sigma} = g^{\lambda\nu} F_{\nu\eta} g^{\eta\sigma}$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \checkmark$$

$$F^{\sigma\tau} = F_{\sigma\tau} \quad \checkmark$$

weiter: $\partial_\mu \frac{1}{2} \varepsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau} \partial_\mu(\partial_\sigma A_\tau - \partial_\tau A_\sigma)$

$$= \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau}(\partial_\mu\partial_\sigma A_\tau - \partial_\mu\partial_\tau A_\sigma)$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\partial_\mu\partial_\sigma A_\tau - \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\partial_\mu\partial_\tau A_\sigma$$

$$\text{—"—} \qquad - \frac{1}{2}\varepsilon^{\mu\nu\tau\sigma}\partial_\mu\partial_\sigma A_\tau$$

$$= \underbrace{\varepsilon^{\mu\nu\sigma\tau}\partial_\mu\partial_\sigma}_{\text{sym / asym}} A_\tau = 0 \quad \checkmark$$

hingegen: $\partial^\mu F^{\nu\sigma} + \partial^\sigma F^{\mu\nu} + \partial^\nu F^{\sigma\mu} = 0$

wichtig!! mit normalem!!

Kontinuitätsgleichung / Ladungserhaltung

$$\operatorname{div} \vec{j} + \partial_t \rho = 0$$

$$\partial_\mu j^\mu = \partial_t \rho + (\vec{\nabla} \cdot \vec{j}) = 0$$

normale Ableitung hat untere Idx
normale Vekt hat obere

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$\underbrace{\partial_\nu \partial_\mu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{asymm}} = \frac{4\pi}{c} \partial_\nu j^\nu = 0 \qquad \text{nötig f. Konsist.}$$

Ladungserh.

$$\int_V d^4x\, \partial^\mu = \int_{\partial V} d^3\sigma^\mu \dots$$

beide raumartig — Seiten HF liegt im Rauml. ∞

$$\int_V d^4x\, \underbrace{\partial^\mu j_\mu}_{\emptyset} = \int_\Sigma d^3\sigma^\mu j_\mu - \int_{\Sigma'} d^3\sigma^\mu j_\mu + \int_\Gamma d^3\sigma^\mu j_\mu = 0$$

j_μ räuml. lok. $\rightarrow$ seitl. Hyperebenen kein Beitrag, da im ∞

und $\int$ $x^0 = ct = \text{const}$ // spezielles Volumen

$$\int_\Sigma d^3\sigma^\mu j_\mu = \int_{x^0=\text{const}} d^3r\, j_0(x^0, \vec{r})$$ ist Zeitunabh.

$$\rightarrow Q = \int d^3r\, \rho \quad \text{// zeitunabh!}$$

$$d^3\sigma^\mu = (dx^1 dx^2 dx^3,\ dx^2 dx^3 dx^0,\ dx^3 dx^0 dx^1,\ dx_0 dx^1 dx^2)$$

$x^0 = \text{const}$

$$\rightarrow d^3\sigma^\mu = (d^3r, 0, 0, 0) \dots$$

$$\rightarrow \underbrace{\int_\Sigma d^3\sigma^0 j_0}_{\int d^3r \rho(\vec{r}, t_1)} = \int_{\Sigma'} d^3\sigma^0 j_0$$

$$\int d^3r\, \rho(\vec{r}, t_1) = \int d^3r\, \rho(\vec{r}, t_2)$$

Transformationseigenschaften

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad A^\mu \rightarrow A'^\mu + \partial^\mu \lambda$$

ändert $F^{\mu\nu}$ nicht, da $\partial^\mu \partial^\nu \lambda - \partial^\nu \partial^\mu \lambda = \emptyset$

$\rightarrow$ Ausdr. von Gesetz mit $F^{\mu\nu}$ macht es Eichinvar.

Lorenzeich

$$\left(\operatorname{div} A + \frac{1}{c}\partial_t \phi\right) = \emptyset$$

$$\partial_\mu A^\mu_L = \emptyset$$

$$\longrightarrow \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$\partial_\mu\big(\partial^\mu A^\nu - \underbrace{\partial^\nu A^\mu\big)}_{\text{verschw}} = \frac{4\pi}{c} j^\nu$$

$$\longrightarrow \quad \partial_\mu \partial^\mu A^\nu_L = \frac{4\pi}{c} j^\nu \qquad \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lorenz – Trafo

$$x'^\mu = \Lambda^\mu{}_\eta \, x^\eta \qquad F'^{\mu\nu} = \Lambda^\mu{}_\sigma \, F^{\sigma\tau} \, \Lambda^\nu{}_\tau \quad \checkmark$$

$$\Longrightarrow \boxed{\begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y' \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & \emptyset \end{pmatrix}} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & \emptyset \end{pmatrix}$$

$1: \; -\beta\gamma E_x \qquad 2: \; -\gamma E_x \qquad 3: \; -\gamma E_y + \beta\gamma B_z$

$5: \; \gamma E_x \qquad 6: \; \beta\gamma E_x \qquad 7: \; \beta\gamma E_y - \gamma B_z$

$4: \; -\gamma E_z - \beta\gamma B_z$

$8: \; \beta\gamma E_z + \gamma B_y$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} A & B & 3 & 4 \\ C & D & 7 & 8 \\ E & F & 11 & 12 \\ G & H & 15 & 16 \end{pmatrix}$$

$A = \gamma 1 - \beta\gamma 2$

$C = \gamma 5 - \beta\gamma 6$

$E = \gamma 9 - \beta\gamma 10$

$G = \gamma 13 - \beta\gamma 14$

$$B = -\beta\gamma\, 1 + \gamma\, 2 \qquad E_x' = C = \gamma^2 E_x - \beta^2\gamma^2 E_x$$
$$D = -\beta\gamma\, 5 + \gamma\, 6 \qquad\qquad = \gamma^2 \underbrace{(1-\beta^2)}_{\gamma^{-2}} E_x$$
$$F = -\beta\gamma\, 9 + \gamma\, 10$$
$$H = -\beta\gamma\, 13 + \gamma\, 14$$

✓

$$B_x' = B_x$$

$$E_y' = E = \gamma E_y - \beta\gamma B_z = \gamma(E_y - \beta B_z)$$
$$E_z' = G = \gamma(E_z + \beta B_y)$$

$$B_y' = \gamma(B_y + \beta E_z)$$
$$B_z' = \gamma(B_z - \beta E_y)$$ ✓

Invarianten d FT

$$F^{\mu\nu} F_{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$$
$$F^{\mu\nu} \hat{F}_{\mu\nu} = -4\, \vec{E}\cdot\vec{B}$$
$$\rightarrow \vec{E}^2 - \vec{B}^2 = \vec{E}'^2 - \vec{B}'^2$$
$$\vec{E}\vec{B} = \vec{E}'\cdot\vec{B}'$$

- Ist $|\vec{E}| > (< =) |\vec{B}|$ dann überall
- Winkel zw $\vec{E}$ & $\vec{B}$ $> (< =) \frac{\pi}{2}$, dann überall
- Verschw $\vec{E}$ od $\vec{B}$ irgendwo, so $\vec{E} \perp \vec{B}$ anderswo
- $\vec{E} \perp \vec{B}$ dann $\exists$ $\vec{E}^2 - \vec{B}^2 \neq 0$ ein BS, wo $\vec{E}$ od $\vec{B}$ $= 0$
- $\vec{E} \perp \vec{B}$ & $\vec{E}^2 - \vec{B}^2 = 0$, dann überall!

Kraft & Energiegleichungen

$$\vec{f} = \rho\vec{E} + \vec{j} \times \vec{B} = \frac{\rho}{c}(c\vec{E} + \vec{v} \times \vec{B})$$

$$\rightarrow f^\mu = \frac{1}{c} F^{\mu\eta} j_\eta \quad \text{mit} \quad f^i = \overbrace{\frac{1}{c} F^{ik} j_k}^{\rho \vec{v}\times\vec{B}} + \overbrace{\frac{1}{c} F^{i0} j_0}^{\rho\vec{E}}$$

$$p = \rho_{\text{ruhe}}\,\gamma \qquad f^i = \frac{1}{c}\,\rho_{\text{ruhe}}\,\gamma\ \vec{v}\times\vec{B} + \dots$$

$$v^i = \gamma\,\tilde{v}$$

$$\to f^i = \frac{1}{c}\,\rho_{\text{ruhe}}\,\cancel{\gamma}\,\frac{v^i}{\cancel{\gamma}} + \dots \quad ✓$$

$$f^0 = \frac{1}{c}F^{0\eta}j_\eta = \frac{1}{c}\vec{E}\vec{j}$$

Elimination d 4er Stromes

$$f^\mu = \frac{1}{c}F^{\mu\nu}j_\nu \qquad \partial^\sigma F_{\sigma\eta} = \frac{4\pi}{c}j_\eta$$

$$\Rightarrow f^\mu = \frac{1}{4\pi}F^{\mu\nu}\partial^\sigma F_{\sigma\nu}$$

$$= \frac{1}{4\pi}\left(\partial^\sigma(F^{\mu\nu}F_{\sigma\nu}) - F_{\sigma\nu}\partial^\sigma F^{\mu\nu}\right)$$

jetzt ∂ rausheben versuchen!

2. Term zuerst 2 Indz tauschen & 1 von und F, dann beide F idx permut., dann mit 1 Vergl. & komb, dann inh. MWG, dann symm Ablt zusammenziehen, dann ρ einfuhren f Ablt → fertig

$$F_{\sigma\nu}\partial^\sigma F^{\mu\nu} = F_{\nu\sigma}\partial^\nu F^{\mu\sigma} = F_{\sigma\nu}\partial^\nu F^{\sigma\mu} = \frac{1}{2}F_{\sigma\nu}(\partial^\sigma F^{\mu\nu} + \partial^\nu F^{\sigma\mu})$$

$$= -\frac{1}{2}F_{\sigma\nu}\,\partial^\mu F^{\nu\sigma} = \frac{1}{2}F_{\sigma\nu}\partial^\mu F^{\sigma\nu} = \frac{1}{4}\partial^\mu(F_{\sigma\nu}F^{\sigma\nu})$$

1. Term noch bearb: $\partial^\sigma(F^{\mu\nu}F_{\sigma\nu})$ Umb d Ind σ & ν

umb. v σ & ν $\partial^\nu(F^{\mu\sigma}F_{\nu\sigma})$ umdrehen

$$\rightarrow \quad -\partial^\nu (F^{\mu\delta} F_{\delta\nu}) \qquad F \text{ hint soll hoch zeigen}$$

$$F_{\delta\nu} = F_\delta{}^\sigma g_{\sigma\nu} \quad \rightarrow -g_{\sigma\nu}\partial^\nu (F^{\mu\delta} F_\delta{}^\sigma)$$

$$= -\partial_\sigma (F^{\mu\delta} F_\delta{}^\sigma) = -\partial_\nu (F^{\mu\delta} F_\delta{}^\nu)$$

✓

& nochmals 2. Term: $\nu \rightarrow \sigma$

$$\frac{1}{4}\partial^\mu (F_{\delta\sigma} F^{\delta\sigma}) = \frac{1}{4} g^{\mu\nu}\partial_\nu (-\text{"}-)$$

$$= \frac{1}{4}\partial_\nu (g^{\mu\nu} F_{\delta\sigma} F^{\delta\sigma})$$

$$\rightarrow \frac{f^\mu}{c} = -\frac{1}{4\pi c}\partial_\nu \left(F^{\mu\delta} F_\delta{}^\nu + \frac{1}{4\pi} g^{\mu\nu} F_{\delta\sigma} F^{\delta\sigma} \right)$$

Energie – Impuls Tensor

$$T^{\mu\nu} = \frac{1}{4\pi c}\left(F^{\mu\delta} F_\delta{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\delta\sigma} F^{\delta\sigma} \right)$$

$$\rightarrow \frac{1}{c} f^\mu = -\partial_\nu T^{\mu\nu}$$

// symmetrisch!!

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{c} w & \frac{1}{c^2}\vec{S} \\ \vec{g} & -\frac{1}{c} T_{ij}^{(M)} \end{pmatrix} \qquad \vec{g} = \frac{1}{c^2}\vec{S}$$

$$\frac{1}{c} w_{em} = \frac{1}{c}\frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$$

$$\vec{g} = \frac{1}{4\pi}(\vec{E} \times \vec{B}) \qquad \vec{S} = \frac{c^2}{4\pi}(\vec{E} \times \vec{B})$$

$$T_{ij}^{(M)} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j + \frac{1}{2}\delta_{ij} (\vec{E}^2 + \vec{B}^2) \right)$$

$T^{\mu\nu}$: Energie – Impuls Tensor

$T_{ij}^{(M)}$: Spannungstensor

Bilanzgleichungen

Zusammenfsg. d Kraftdichte gl.

$$\frac{1}{c} f^\mu = -\partial_\nu T^{\nu\mu} \qquad \& \qquad f^\mu = \frac{1}{c} F^{\mu\nu} j_\nu$$

$$\partial_\nu T^{\nu\mu} + \frac{1}{c^2} F^{\mu\nu} j_\nu = 0$$

$\rightarrow$ Energie & Impulserhaltung

$$\partial_\nu T^{\nu 0} + \frac{1}{c^2} F^{0\nu} j_\nu \quad :$$

$$\frac{1}{c^2}\left(\partial_t w + \vec{\nabla}\vec{S} + \vec{E}\vec{j}\right) = 0$$

$$\partial_\nu T^{\nu i} + \frac{1}{c^2} F^{i\nu} j_\nu = 0 \quad :$$

$$\frac{1}{c}\partial_t \vec{g} - \frac{1}{c}\partial^l T_{li} + \frac{c}{c^2} E_i \varrho + \frac{1}{c}(j \times B)_i = 0$$

$$\frac{1}{c}\left(\partial_t \vec{g} - \partial^l T_{li} + E_i \varrho + (\vec{j} \times B)_i\right) = 0$$

Viererimpuls d EM-Feldes

$$p^\mu_{feld} = \int_\Sigma d\sigma_\nu T^{\nu\mu} = \int d^3r\, T^{0\mu}$$

oberflächen ins Zeithyperebene

$$\frac{d}{dt} p^\mu_{teil} = \int d^3r\, f^\mu$$

aus neuen Kraftdichte gl:

$$\frac{1}{c} f^\mu + \partial_\nu T^{\nu\mu} = 0$$

$$\int d^3r \left(\frac{1}{c} f^\mu + \frac{1}{c}\partial_t T^{0\mu} + \partial_i T^{i\mu}\right) = 0$$

$$\rightarrow \frac{1}{c}\frac{d}{dt}\left(p^\mu_{Teilch} + p^\mu_{Feld}\right) = -\int d^3r\ \partial_i T^{i\mu}$$

BWGL f. gel. Punktteilch.

$$f^\mu = \frac{1}{c} F^{\mu\nu} j_\nu$$

$$\frac{d}{dt}\vec{p} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

Änd. d. Energie: $E = \vec{F}\, d\vec{s}$ // $E = m\gamma c^2$

$$\frac{d}{dt}\vec{F}\, d\vec{s} = \vec{F}\frac{d\vec{s}}{dt} = \vec{F}\vec{v} = q\vec{E}\vec{v}$$

kann man zu

Feldstärketensor

$$\frac{d}{d\tau} p^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$$

// an Kräfte v. äußeren Feld bei.

$$= F^\mu$$

4er Kraft

$$j^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} = \begin{pmatrix} \rho & c \\ \rho & \vec{v} \end{pmatrix} = \rho u^\mu$$

hier nicht mit j^μ, da es eine Dichte ist!

man müsste auch j^μ über Vol $\int$! $\rightarrow q u_\nu$

Relativistische Optik

Ebene Wellen

quellenfrei: $\partial_\mu F^{\mu\nu} = 0$

auf hom MWG $\partial^\mu F^{\sigma\nu} + \partial^\sigma F^{\nu\mu} + \partial^\nu F^{\mu\sigma} = 0$

∂_μ anwenden $\longrightarrow \partial_\mu \partial^\mu F^{\sigma\nu} = 0$ Wellengleichung!!

Ansatz: $F^{\sigma\nu}(\vec{r},t) = f^{\sigma\nu} e^{-ik_\tau x^\tau}$

Bei ebenen Wellen V ist f das $e^{i(\vec{k}\vec{r}-\omega t)}$

$$\partial_\mu \partial^\mu f^{\sigma\nu} e^{-ik_\tau x^\tau} = 0$$

$\rightarrow \quad k_\mu k^\mu = \emptyset \qquad\qquad k^\mu = \begin{pmatrix} k_0 \\ \vec{k} \end{pmatrix} = \begin{pmatrix} \frac{\omega}{c} \\ \vec{k} \end{pmatrix} = \begin{pmatrix} |\vec{k}| \\ \vec{k} \end{pmatrix}$

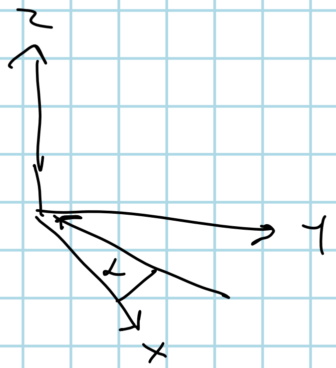
$\rightarrow \quad \frac{\omega^2}{c^2} - \vec{k}^2 = \emptyset \qquad\qquad \omega = c|\vec{k}| \quad \checkmark$

$\rightarrow \quad p^\mu = \hbar k^\mu = \begin{pmatrix} \hbar \frac{\omega}{c} \\ \hbar \vec{k} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$

Doppler Effekt

Frage – Änd einer ebenen Welle bei Red bezo System

Welle in S: $\quad k^\mu = \begin{pmatrix} k \\ -k\cos\alpha \\ -k\sin\alpha \\ \emptyset \end{pmatrix}$ } zum Ursprung



Trafo: $\quad k'^\mu = \Lambda^\mu{}_\eta \, k^\eta$

S' in x-Richtung!

$$\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} k \\ -k\cos\alpha \\ -k\sin\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma k + \beta\gamma k\cos\alpha \\ -\beta\gamma k - \gamma k\cos\alpha \\ -k\sin\alpha \\ 0 \end{pmatrix} = k'^\mu$$

$$= \begin{pmatrix} \gamma k(1 + \beta\cos\alpha) \\ -\gamma k(\beta + \cos\alpha) \\ -k\sin\alpha \\ 0 \end{pmatrix} \qquad \frac{\omega'}{\cancel{c}} = \gamma \frac{\omega}{\cancel{c}}(1 + \beta\cos\alpha) \quad \checkmark$$

Long & transvers. Doppler

Long: $\quad \alpha = 0 \qquad \omega' = \gamma\omega(1+\beta)$

$\qquad\quad \alpha = \pi \qquad \omega' = \gamma\omega(1-\beta)$

transv: $\quad \alpha = \frac{\pi}{2} \qquad \omega' = \gamma\omega \qquad$ // nicht klassisch!

Abberation

Richtungsänd einer Welle

$$k'^{\mu} = \begin{pmatrix} k' \\ -k'\cos\alpha' \\ -k'\sin\alpha' \\ 0 \end{pmatrix}$$

$$k' = \gamma k (1 + \beta\cos\alpha)$$
$$-k'\cos\alpha' = -\gamma k (\beta + \cos\alpha)$$
$$-k'\sin\alpha' = -k\sin\alpha$$

$$-\cancel{\gamma k}(1+\beta\cos\alpha)\cos\alpha' = -\cancel{\gamma k}(\beta + \cos\alpha)$$

$$\longrightarrow \cos\alpha' = \frac{\beta + \cos\alpha}{1+\beta\cos\alpha}$$

$$\sin\alpha' = \frac{\sin\alpha}{\gamma(1+\beta\cos\alpha)} \quad \checkmark$$

$$\cos\alpha = \frac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}$$

$$\cos\alpha' = \frac{1 - tg^2\frac{\alpha'}{2}}{1 + tg^2\frac{\alpha'}{2}} = \frac{\beta + \cos\alpha}{1 + \beta\cos\alpha} = \frac{\beta + \frac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}}{1 + \beta\frac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}}$$

$$= \frac{\beta\left(1 + tg^2\frac{\alpha}{2}\right) + 1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2} + \beta\left(1 - tg^2\frac{\alpha}{2}\right)} = \frac{1 - tg^2\frac{\alpha}{2} + \beta\left(1 + tg^2\frac{\alpha}{2}\right)}{1 + tg^2\frac{\alpha}{2} + \beta\left(1 - tg^2\frac{\alpha}{2}\right)}$$

NR: $1 - tg^2\frac{\alpha}{2} + \beta + \beta tg^2\frac{\alpha}{2}$

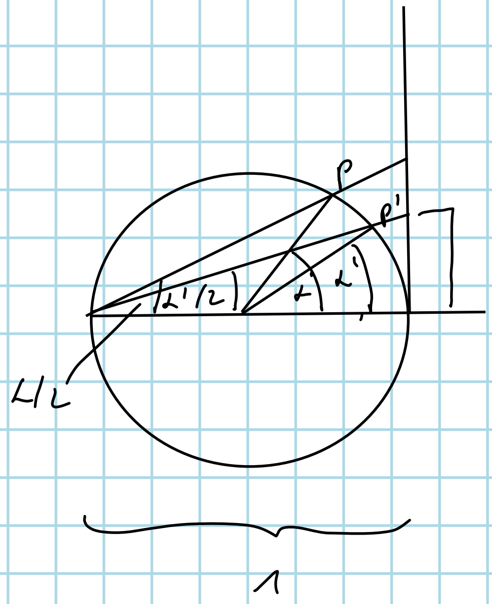
$$\frac{1 + \beta - tg^2\frac{\alpha}{2}(1-\beta)}{1 + \beta + tg^2\frac{\alpha}{2}(1-\beta)}$$

NRZ:

$$\Rightarrow \frac{1 - \left(tg^2\frac{\alpha}{2}\right)\frac{1-\beta}{1+\beta}}{1 + \left(tg^2\frac{\alpha}{2}\right)\frac{1-\beta}{1+\beta}} \quad \checkmark$$

$$\Rightarrow tg\frac{\alpha'}{2} = \sqrt{\frac{1-\beta}{1+\beta}}\, tg\frac{\alpha}{2} \quad \checkmark$$

$$\longrightarrow \alpha > \alpha'$$



$\mathrm{tg}\,\frac{\alpha'}{2}$

$$\mathrm{tg}\,\frac{\alpha}{2} = \sqrt{\frac{1+n}{1-n}}\,\mathrm{tg}\,\frac{\alpha'}{2}$$

Penrose Bez

Hamilton Formalismus

Newton $\vec{F} = m\,\frac{d\vec{v}}{dt}$

- D'Alembertsches Prinzip
- Lagrange Gleichp.
- Hamilton Prinzip
- Hamilton Gl.

D' Alembert:

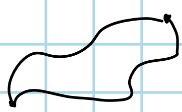
$$\sum_i \left(\vec{F}_i - m_i \ddot{\vec{r}}_i\right) \cdot \delta \vec{r}_i = 0$$

bei virtueller Verschiebung wird keine Arbeit verrichtet

i-tes Teilchen

Summenterme müssen nicht einzeln verschw., da Verrück. d. Zwangsbed. abh. sein können

Lagrange Gleichungen



weil an Endpkt

$$L = L(q_i, \dot{q}_i, t) \qquad \delta \int_{t_0}^{t_1} L\,dt = 0 \qquad \delta(t_0) = \delta(t_1) = 0$$

$L = T - V$

$$\sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \underbrace{\frac{\partial L}{\partial \dot q_i} \underbrace{\delta \dot q_i}_{\frac{d}{dt}\delta q_i}}_{-\frac{d}{dt}\left(\frac{\partial L}{\dot q_i}\right)\delta q_i} \right) = \emptyset$$

Randterm fällt weg

$L = T - V$

$$\rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \emptyset$$

gilt ganz allg. f. konservatives System

Verallg. Kräfte $F_{q_i} = \sum_t \vec F_t \frac{\partial \vec r_t}{\partial q_i}$

$$\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot q_i} = \frac{\delta A}{\delta q_i} = Q_i$$

$\delta A = \sum Q_i \, \delta q_i$

$\delta A = \sum \vec F_t \, \delta \vec r_t$

$$/\cdot(-1) \quad \frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} = \underbrace{-\frac{\partial V}{\partial q_i}}_{Q_i}$$

$$\frac{\delta A}{\delta q_i} = \sum_t \frac{\vec F_t \, \partial \vec r_t}{\partial q_i}$$

auch Kräfte die keine Potentiale

$$T = \sum_t \frac{m_t \vec v_t^2}{2}$$

dann $\quad$ — " — $+ \frac{\partial V}{\partial q_i} = Q_i$ ✓

Extremalprinzip f. d. Wirkung

$$S = \int_{t_1}^{t_2} L\, dt \quad \overline{L}(q, \dot q, t)$$

$\delta S = \emptyset \qquad$ mit $\quad \delta q_i(t_1) = \delta q_i(t_2) = \emptyset$

Ganze Herleitung nur aus Extremalprinzip, Herleitung aus Lagrange unbekannt...

Hamilton Gleichungen

aus $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = \emptyset$ $\qquad \dot p_i = \frac{\partial L}{\partial q_i}$

mit $p_i = \frac{\partial L}{\partial \dot q_i}$

$$H = \sum_i p_i \dot q_i - L = H(q, p, t)$$

$$dH = \sum_i \cancel{p_i\, d\dot q_i} + \dot q_i\, dp_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot p_i} dq_i - \cancel{\underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i} d\dot q_i} - \frac{\partial L}{\partial t} dt$$

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

$$\rightarrow \quad \dot q_i = \frac{\partial H}{\partial p_i} \qquad \dot p_i = -\frac{\partial H}{\partial q_i}$$

BWGL (Hamilton Gl)

Euler Lagrange GL : i DG 2 Ord.

Hamilton GL : 2i DG 1. Ord.

$$H = T + V$$

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}}_{-\dot p} \underbrace{\frac{dq}{dt}}_{\dot q} + \underbrace{\frac{\partial H}{\partial p}}_{\dot q} \underbrace{\frac{dp}{dt}}_{\dot p} + \frac{\partial H}{\partial t}$$

wenn $= 0$ dann $\frac{dH}{dt} = \emptyset$

dann H Erhaltungsgröße

Hamiltonfkt. eines freien Pkt Teilchen $\quad // \; V = \emptyset$

$$L = T = \sum_i \frac{m \dot q_i^2}{2} \qquad p_i = \frac{\partial L}{\partial \dot q} = m \dot q_i$$

$$H = \sum_i p_i \dot q_i - L = \sum_i p_i \frac{p_i}{m} - \frac{p_i^2}{2m}$$

$$H = \sum_i \frac{p_i^2}{2m} \qquad \dot q_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot p_i = -\frac{\partial H}{\partial q_i} = \varnothing$$

$$= \frac{p_i}{m} \quad ✓$$

Relativistische Hamilton Fkt.

herleit über Wirkung $\qquad d\tau = \frac{dt}{\gamma}$

$$S = \int_{t_1}^{t_2} L \, dt = \int_{\tau_1}^{\tau_2} \gamma L \, d\tau \qquad S \text{ ist invar.}$$

$\rightarrow$ γL ist invariant !!

Bildung: $\quad m \; \& \; u^\nu \qquad u^\mu u_\mu = c^2$

$$\gamma L = -mc^2$$

$$\rightarrow L_{pt} = -\frac{mc^2}{\gamma} = -mc^2 \sqrt{1-\frac{\vec u^2}{c^2}}$$

$$p_i = \frac{\partial L}{\partial \dot q_i} \qquad p_i = -mc^2 \frac{1}{\cancel{2}} (-\text{"}-)^{-\frac{1}{2}} \cdot \frac{-\cancel{2} u_i}{c^2}$$

$$= m\gamma u_i \quad ✓$$

$$\rightarrow H_{pt} = \sum_i p_i \dot q_i - L = \underbrace{p_i \frac{p_i}{m\gamma}}_{\frac{m^2\gamma^2 u^2}{m\gamma}} + \frac{mc^2}{\gamma}$$

$$= m\gamma u^2 + \frac{mc^2}{\gamma} = m\gamma c^2 \left(\frac{u^2}{c^2} + \gamma^{-2}\right)$$

$$= m\gamma c^2 \left(\cancel{\frac{u^2}{c^2}} + 1 - \cancel{\frac{u^2}{c^2}}\right) ✓$$

$$= m\gamma c^2$$

BWGL eines freien PktT.

$$\dot x_i = \frac{\partial H}{\partial p_i} \qquad \dot p_i = \frac{\partial H}{\partial x_i}$$

$$H = m\gamma c^2 \qquad p^2 = m^2\gamma^2 u^2$$

$$\gamma^{-2} p^2 = m^2 u^2$$

$$\left(1 - \frac{u^2}{c^2}\right) p^2 = m^2 u^2$$

$$p^2 = m^2 u^2 + \frac{p^2 u^2}{c^2}$$

$$p^2 = u^2 \left(m^2 + \frac{p^2}{c^2} \right)$$

$$\rightarrow \frac{u^2}{c^2} = \frac{p^2}{c^2\left(\frac{p^2}{c^2} + m^2\right)} = \frac{p^2}{p^2 + m^2c^2} \checkmark$$

$$1 - \frac{u^2}{c^2} = \frac{\cancel{p^2} + m^2c^2 \cancel{- p^2}}{N} = \frac{m^2c^2}{p^2 + m^2c^2} \; \ldots$$

$$\Rightarrow \gamma^{-1} = \frac{mc}{\sqrt{p^2 + m^2c^2}}$$

$$H_{ft} = \gamma m c^2 = \sqrt{p^2 + m^2c^2} \cdot c = \sqrt{m^2c^4 + p^2c^2}$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{\partial H}{\partial x_i} = \emptyset$$

$$\frac{\partial H}{\partial p_i} = \frac{1}{\cancel{2}} (\qquad)^{-\frac{1}{2}} \cdot c^2 \cancel{2} p_i = \frac{c^2 p_i}{\sqrt{m^2c^4 + p^2c^2}}$$

$$= \frac{c p_i}{\sqrt{m^2c^2 + p^2}} = \frac{p_i}{\gamma m} \checkmark$$

Freie Felder

anstelle eines Pkts. mit Weltlinie ein Feld ϕ im ganzen Raum

Dichte invariant!

$$S(\phi, \partial^\nu \phi) = \int_{t_1}^{t_2} L(t) = \frac{1}{c} \int_\Omega d^4x \, \mathcal{L}(\phi(x^\mu), \partial^\nu \phi(x^\mu))$$

$$\delta S = \delta \int_{t_1}^{t_2} dt\, L(t) = \delta \int_{t_1}^{t_2} dt \int d^3r\, \mathcal{L}\left(\phi(\vec{r},t), \vec{\nabla}\phi(\vec{r},t), \dot{\phi}(\vec{r},t)\right) = 0$$

$$\delta\phi(\vec{r},t_1) = \delta\phi(\vec{r},t_2) = 0$$

wieder Euler Lagrange

$$\mathcal{L} = \mathcal{L}(\phi, \partial^\nu \phi)$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial^\nu \phi)}\overset{\partial^\nu(\delta\phi)}{\delta(\partial^\nu\phi)}}_{-\partial^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu \mathcal{L})}\right)\delta\phi} = 0$$

$$\delta\mathcal{L} = \delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi)}\right)\right)$$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi)}\right) = 0$$

Statt ϕ z.B. A^μ

$$\mathcal{L} = \mathcal{L}(A^\mu, \partial^\nu A^\mu)$$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial A^\mu} - \partial^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu A^\mu)}\right) = 0$$

$\mathcal{L}$ ist nicht eindeutig, da

$$\mathcal{L} \rightarrow \mathcal{L} + \partial^\nu B_\nu$$

Unbestimmtheit: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$

$$L = L + \frac{d}{dt} f(q)$$

$$\delta S = \delta\left(\int_{t_1}^{t_2}\left(L + \frac{d}{dt} f\right)\right) = \delta\int_{t_1}^{t_2} L + \delta\int_{t_1}^{t_2}\frac{d}{dt} f = 0$$

$\delta f \Big|_{t_1}^{t_2}$ kein Beitrag

Impulsdichte analog zu $p_i = \frac{\partial L}{\partial \dot q_i}$

$$\pi(\vec r, t) = \frac{\partial \mathcal{L}}{\partial \dot\phi}$$

Hamiltondichte

$$\mathcal{H} = \pi \dot\phi - \mathcal{L}$$

$$H = \int d^3r \, \mathcal{H}(\phi, \pi)$$

Lagrangedichte aus Feldstärketensor bilden. 1. ist Lorentzskalar, 2. ist Pseudoskalar

$$F_{\mu\nu} F^{\mu\nu} = -2(\vec E^2 - \vec B^2)$$

$$\mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi}(\vec E^2 - \vec B^2)$$

$\frac{1}{8\pi}$ ist gl. Faktor wie bei Energiedichte!

$$\mathcal{L}(A^\mu, \partial^\nu A^\mu) = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial A^\mu}}_{\emptyset} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)}$$

$$\mathcal{L} = -\frac{1}{16\pi} g_{\mu\alpha} g_{\nu\beta} \overbrace{(\partial^\alpha A^\beta - \partial^\beta A^\alpha)}^{F^{\alpha\beta}} \overbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}^{F^{\mu\nu}}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\sigma A^\tau)} = -\frac{1}{16\pi} g_{\mu\alpha} g_{\nu\beta} \Big((\delta^\alpha_\sigma \delta^\beta_\tau - \delta^\beta_\sigma \delta^\alpha_\tau) F^{\mu\nu} + F^{\alpha\beta}(\delta^\mu_\sigma \delta^\nu_\tau - \delta^\nu_\sigma \delta^\mu_\tau) \Big)$$

$$= -\frac{1}{16\pi}\Big(g_{\mu\sigma} g_{\nu\tau} F^{\mu\nu} - g_{\mu\tau} g_{\nu\sigma} F^{\mu\nu} + g_{\alpha\sigma} g_{\beta\tau} F^{\alpha\beta} - g_{\alpha\tau} g_{\beta\sigma} F^{\alpha\beta} \Big)$$

$$= -\frac{1}{16\pi}(4 F_{\sigma\tau}) = -\frac{1}{4\pi} F_{\sigma\tau}$$

BWGL:

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \underbrace{\frac{1}{4\pi} \partial^\nu F_{\nu\mu}}_{\text{inh. MWG f. Quellenfr. Geb.}} = \emptyset$$

Kanonischer Impuls (Konjugierte)

$$p_i = \frac{\partial L}{\partial \dot q_i}$$

$$\pi_\lambda = \frac{\partial \mathcal{L}}{c\,\partial(\partial^0 A^\mu)} = \frac{\partial \mathcal{L}}{\partial \dot A^\mu(\vec r, t)} = -\frac{1}{c}\frac{1}{4\pi} F_{0\mu}$$

$$\pi_0 = c_1 \cdot F_{00} = \emptyset$$

$$\pi_i = -\frac{1}{4\pi c} F_{0i} = E_i \frac{1}{4\pi c}$$

$F^{\mu\nu}$ hat andere VZ als $F_{\mu\nu}$!

$$g_{\mu\sigma} F^{\mu\nu} g_{\nu\eta} = F_{\sigma\eta}$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & & & \\ E_y & & & \\ E_z & & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & & & \\ -E_y & & \text{old} & \\ -E_z & & & \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & & & \\ -E_y & & \text{old} & \\ -E_z & & & \end{pmatrix}$$

$\rightarrow F_{ij} = F^{ij}$

$F^{0\mu} = -F_{0\mu}$

$F^{\mu 0} = -F_{\mu 0}$

$$\mathcal{H} = c\pi_\lambda \partial^0 A^\lambda - \mathcal{L} \qquad H = \int d^3r\, \mathcal{H}$$

$$\mathcal{H} = -\frac{1}{4\pi c} E_i \underbrace{\frac{\partial A_i}{\partial t}}_{-c\vec E - c\,\mathrm{grad}\,\phi} - \frac{1}{8\pi}(\vec E^2 - \vec B^2)$$

hier darf man nicht mehr E einsetzen, da keine Statik!

$$= \frac{1}{4\pi}(+E_i E_i + E_i \partial_i \phi) - \frac{1}{8\pi}(\vec E^2 - \vec B^2)$$

$$= \frac{1}{8\pi}\underbrace{(2E_i^2 - \vec E^2 + \vec B^2)}_{E^2 + B^2} + \frac{1}{4\pi}\underbrace{E_i \partial_i \phi}_{\partial_i(E_i \phi)}$$

// wegen quellenfrei

$$H = \int d^3r \, \mathcal{H} = \int d^3r \, w$$

EM Wechselwirkung

zw. T & Feld

Lagrange fkt:

$$\mathcal{L}_{ww} = -\frac{1}{c} A^\mu j_\mu$$

$(\vec{r},t)$

$\frac{1}{c}$ ist Faktor wie bei MWG

$$= -\frac{1}{c}\left(\phi \, c\rho - \vec{A}\vec{j}\right)$$

$$= -\phi\rho + \vec{A}\vec{j}$$

Punktteilch.:

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}(t))$$

$$j(\vec{r},t) = q v\,\delta(\vec{r} - \vec{r}(t))$$

$$L_{ww}(t) = \int d^3r \, \mathcal{L}_{ww}(\vec{r},t) = -q\,\phi(\vec{r}(t),t) + \vec{A}(\vec{r},t)\frac{q}{c}\vec{u}(t)$$

kein Lorenz-Skalar

$\gamma L : L_s$?

L_{ww} nicht eich invar, jedoch $L_{ww} + L_{em}$ schon!

BWGL d. em Felder

$$\mathcal{L}(x) = \mathcal{L}_{em}(x) + \mathcal{L}_{ww}(x) = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A^\mu j_\mu$$

aus

$$\frac{\partial\mathcal{L}}{\partial(A^\sigma)} - \partial^\delta \frac{\partial\mathcal{L}}{\partial(\partial^\delta A^\sigma)} = -\frac{1}{c}\underbrace{\delta^\mu_\sigma j_\mu}_{j_\sigma} + \frac{1}{4\pi}\partial^\delta F_{\delta\sigma}$$

$\rightarrow$ inh MWG

hom MWG sind gleich

Impuls von

$$\Pi_\delta = \frac{\partial\mathcal{L}}{c\,\partial(\partial^0 A^\delta)} = \frac{\partial\mathcal{L}}{\partial\dot{A}} = -\frac{1}{c\,4\pi}F_{0\delta} \quad // \text{gleich}$$

BWGL d. Punktteilchens

$$L(\vec{r},\vec{v},t) = L_{pt} + L_{ww} = \overbrace{-mc^2\sqrt{1-\frac{u^2}{c^2}}}^{-\frac{mc^2}{\gamma}} - q\phi + \frac{q}{c}\vec{A}\vec{u}$$

$$p_i = \frac{\partial L}{\partial \vec{v}} = \underbrace{m\gamma u_i}_{\text{mech. Impuls}} + \frac{q}{c}A_i$$

$$H(\vec{r},\vec{P},t) = \vec{P}\vec{u} - L(\vec{r},\vec{u},t)$$

$$= \gamma m u^2 + \cancel{\frac{q}{c}\vec{A}\vec{u}} + \frac{mc^2}{\gamma} + q\phi - \cancel{\frac{q}{c}\vec{A}\vec{u}}$$

$$= \gamma m c^2 \underbrace{\left(\frac{u^2}{c^2} + \frac{1}{\gamma^2}\right)}_{\cancel{\frac{u^2}{c^2}} + 1 - \cancel{\frac{u^2}{c^2}}} + q\phi = \gamma m c^2 + q\phi$$

kanonische Form:

$$\left(p_i - \frac{q}{c}A_i\right)^2 = m^2\gamma^2u^2$$

$$\left(\quad - \quad\right)^2\left(1 - \frac{u^2}{c^2}\right) = m^2u^2$$

$$\left(\quad\right)^2 - \left(\quad\right)^2\frac{u^2}{c^2} = m^2u^2$$

$$\left(\quad\right)^2 = \left(\quad\right)^2\frac{u^2}{c^2} + m^2u^2$$

$$\left(\quad\right)^2 = u^2\left(\frac{\left(\quad\right)^2}{c^2} + m^2\right)$$

$$\rightarrow \frac{u^2}{c^2} = \frac{\left(\quad\right)^2}{\left(\quad\right)^2 + m^2c^2}$$

$$1 - \frac{u^2}{c^2} = \frac{\cancel{\left(\quad\right)^2} + m^2c^2 - \cancel{\left(\quad\right)^2}}{\left(\quad\right)^2 + m^2c^2}$$

$$\gamma^{-1} = \frac{mc}{\sqrt{\left(\quad\right)^2 + m^2c^2}}$$

$$H = \gamma m c^2 + q\phi = \frac{\sqrt{(\quad)^2 + m^2c^2}\,\cancel{m}\,c^2}{\cancel{m}\,\cancel{c}} + q\phi$$

$$= \sqrt{(P_i - \tfrac{q}{c}A_i)^2 c^2 + m^2c^4} + q\phi$$

Reihe: $H = mc^2 + \frac{(P_i - \frac{q}{c}A_i)^2}{2m} + \dots + q\phi$

Bewegungsgl.: $\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$

$$u_i = \frac{\partial H}{\partial P_i} = \frac{1}{\cancel{2}}\left(\dots\right)^{-\frac{1}{2}} \cdot \cancel{2}\left(\dots\right) \cdot c^2$$ — nach Ableitung

$$= \frac{1}{c\sqrt{(\quad)^2 + m^2c^2}}\left(\quad\right)c^2$$

$$\boxed{\gamma = \frac{\sqrt{\quad}}{mc}} = \frac{c^2}{mc^2\gamma}\left(P_i - \tfrac{q}{c}A_i\right)$$

kommt gleich an $\frac{\partial L}{\partial u_i}$ raus !!

$$\frac{d}{dt}P_i = -\frac{\partial H}{\partial x_i} = \frac{1}{\cancel{m\gamma c^2}}c^2\underbrace{\left(P_j - \tfrac{q}{c}A_j\right)}_{\cancel{\gamma m}u_j}\frac{q}{c}\frac{\partial A_j}{\partial x_i} - q\frac{\partial\phi}{\partial x_i} =$$

$$= \frac{q}{c}u_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial\phi}{\partial x_i}$$

$$\frac{d}{dt}P_i = \frac{d}{dt}\left(p_i + \frac{q}{c}A_i\right) = \frac{d}{dt}p_i + \frac{q}{c}\frac{\partial}{\partial t}A_i + \frac{q}{c}\frac{\partial A_i}{\partial x_j}\underbrace{\frac{dx_j}{dt}}_{u_j}$$

Zusammenfassen

$$\frac{d}{dt}p_i = q\left(-\frac{\partial\phi}{\partial x_i} - \frac{1}{c}\partial_t A\right) + \frac{q}{c}\underbrace{u_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)}_{\vec{u}\times\mathrm{rot}\vec{A}}$$

$$= qE_i + \frac{q}{c}(\vec{v}\times\vec{B})$$

Edyn in Materie

Aufteilung der Quellen in

$$j^\mu_{frei} + j^\mu_{mat} = j^\mu_{ges}$$

exp. steuerbar: Ladung verteilen & Strom

nicht steuerbar: Pol. v. Atomen, Kreisströme, magn. Momente von Kernen

MWG linear → superponierbar

$$\partial_\mu F^{\mu\nu}_{frei} + \partial_\mu F^{\mu\nu}_{mat} = \partial_\mu F^{\mu\nu}_{ges} = \frac{4\pi}{c} j^\nu_{ges}$$

$$\partial_\mu \hat{F}^{\mu\nu}_{frei} + \partial_\mu \hat{F}^{\mu\nu}_{mat} = \partial_\mu \hat{F}^{\mu\nu}_{ges} = \emptyset$$

zB: $\partial_\mu F^{\mu\nu}_{ges} = \frac{4\pi}{c}\left(j^\nu_{frei} + j^\nu_{mat}\right)$ machen!

Mittelwertbildung — Ausm. d. atomistischen Struktur

$$\langle F^{\mu\nu}_{frei,mat}(\vec{r},t)\rangle = \int d^3r' \, f(\vec{r}\,') \, F^{\mu\nu}_{frei,mat}(\vec{r}+\vec{r}\,',t)$$

$$\langle j^\mu_{frei,mat}(\vec{r},t)\rangle = \int d^3r' \, f(\vec{r}\,') \, j^\mu_{frei,mat}(\vec{r}+\vec{r}\,',t)$$

$$\int d^3r' \, f(\vec{r}\,') = 1$$

MW ist mit Diffops vertauschbar →

& $j^\mu := \langle j^\mu_{frei}\rangle$ $\quad F^{\mu\nu} := \langle F^{\mu\nu}_{ges}\rangle$

$$\partial_\mu \langle F^{\mu\nu}_{frei}\rangle = \frac{4\pi}{c} j^\nu + \partial_\mu \langle F^{\mu\nu}_{mat}\rangle = \frac{4\pi}{c} j^\nu_{mat} = \partial_\mu F^{\mu\nu} = \langle j^\nu_{ges}\rangle$$

$$\partial_\mu \langle \hat{F}^{\mu\nu}_{frei}\rangle = \emptyset + \partial_\mu \langle \hat{F}^{\mu\nu}_{mat}\rangle = \emptyset = \partial_\mu \hat{F}^{\mu\nu} = \emptyset$$

// gebundene Quellen werden d. Rückw. von Gesamtfeld geändert!

Modifikation d. Teilfelder

Neuaufteilung v. $F^{\mu\nu} = \langle F_{frei}^{\mu\nu} \rangle + \langle F_{mat}^{\mu\nu} \rangle$

in $\underbrace{F^{\mu\nu}}_{\vec{E}, \vec{B}} = \underbrace{H^{\mu\nu}}_{\vec{D}, \vec{H}} + 4\pi \underbrace{M^{\mu\nu}}_{-\vec{P}, \vec{M}}$ sodass $M^{\mu\nu}$ außerhalb d. Mat. $= \emptyset$ ist

el. Polar. / magnetis.

$\langle F_{mat}^{\mu\nu} \rangle$ ist es nicht!

& $H^{\mu\nu}$ selbe inh. Gl. wie $F_{frei}^{\mu\nu}$ hat

$$\partial_\mu H^{\mu\nu} = \frac{4\pi}{c} j^\nu \longrightarrow \partial_\mu \left(H^{\mu\nu} - \langle F_{frei}^{\mu\nu} \rangle \right) = \emptyset$$

Erreichen d. Umformungsgröße $\quad F_{hilf}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Lsg: $\quad \underline{H^{\mu\nu} = \langle F_{frei}^{\mu\nu} \rangle + \hat{F}_{hilf}^{\mu\nu}}$ — dualer Tensor liefert kein MWG → kein Beitrag

$$\partial_\mu \langle F_{frei}^{\mu\nu} \rangle - \partial_\mu \langle -\text{"}- \rangle + \underbrace{\partial_\mu \hat{F}_{hilf}^{\mu\nu}}_{\emptyset} = \emptyset$$

& $\quad \underline{4\pi M^{\mu\nu} = \langle F_{mat}^{\mu\nu} \rangle - \hat{F}_{hilf}^{\mu\nu}}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\hat{F}_{hilf}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

hier sind E & B nur Platzhalter!!

3er Notation aus $\quad H^{\mu\nu} = \langle F_{frei}^{\mu\nu} \rangle + \hat{F}_{hilf}^{\mu\nu}$

$$H^{i0} = D^i = \langle E_{frei}^i \rangle + \underbrace{\hat{F}_{hilf}^{i0}}_{\text{rot}\,\vec{A}}$$

$$4\pi M^{i0} = -4\pi P^i = \langle E_{mat}^i \rangle - \underbrace{\hat{F}_{hilf}^{i0}}_{\text{rot}\,\tilde{A}}$$

siehe innere Komp. von $H^{\mu\nu}$ für folgende Abltg. od. akzeptiere

$$\rightarrow H^i = \langle B^i_{frei} \rangle + \nabla\phi + \frac{1}{c}\partial_t A^i$$

$$4\pi M^i = \langle B^i_{mat} \rangle - \nabla\phi - \frac{1}{c}\partial_t A^i$$

in inneren Komp. steht ein $-\vec{E}$ beim dualen

$$\longrightarrow \quad D^i - 4\pi P^i = \langle E^i_{frei} \rangle + \langle E^i_{mat} \rangle = E^i$$

$$H^i + 4\pi M^i = B^i$$

$\longrightarrow$ Makroskopische MWG

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\left(j^\nu + \langle j^\nu_{mat} \rangle\right)$$

$$\partial_\mu H^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$\partial_\mu 4\pi M^{\mu\nu} = \frac{4\pi}{c}\langle j^\nu_{mat} \rangle$$

inh. MWG

$\partial_\mu \hat{F}^{\mu\nu} = \emptyset$ hom. MWG f. F bleiben gleich

$$\partial_\mu \hat{H}^{\mu\nu} = \underbrace{\partial_\mu \langle \hat{F}^{\mu\nu}_{frei} \rangle}_{\emptyset} - \partial_\mu F^{\mu\nu}_{hilf} \neq \emptyset$$

$$\partial_\mu 4\pi \hat{M}^{\mu\nu} = \partial_\mu F^{\mu\nu}_{hilf} \neq \emptyset$$

hom. f. H & M noch nicht betrachtet!

Änderungen d. MWG je nur f. M & H

enthalten Umformungsgr. ϕ & $\vec{A}$ welche indirekt über $\vec{P}$ & $\vec{M}$ festgelegt werden

inh. Gl. mit Materie Termen:

$$\partial_i(-4\pi P_i) = 4\pi\langle \rho_{mat} \rangle \quad \rho_P$$

$$\text{rot}\left(4\pi\vec{M}\right) = \frac{4\pi}{c}\underbrace{\langle \vec{j}_{mat} \rangle} + \frac{1}{c}\partial_t\left(-4\pi\vec{P}\right)$$

$$= \frac{4\pi}{c}\underbrace{\left(\langle \vec{j}_{mat} \rangle - \partial_t\vec{P}\right)}_{\vec{j}_M} \qquad \partial_t\vec{P} = \vec{j}_P$$

$\rightarrow \quad \mathrm{div}\, \vec{P} = -\rho_P$

$\mathrm{rot}\, \vec{M} = \frac{1}{c} \vec{j}_M$

$\partial_\mu j^\mu_{\substack{frei\\ mat}} = 0$

$\mathrm{div}\, \partial_t \vec{P} = -\partial_t \rho_P$

$\mathrm{div}\, \vec{j}_P + \partial_t \rho_P = 0$ ✓

aber: $\underbrace{\mathrm{div}\, \mathrm{rot}\, \vec{M}}_{0} = \frac{1}{c} \mathrm{div}\, \vec{j}_M$

$\rightarrow \mathrm{div}\, \vec{j}_M = 0$! // Magnetisierungsstrom ist Quellenfrei

Festlegung von $\vec{P}$ & $\vec{M}$

Innerhalb sind sie Fkt von Gesamtfeldern

außerhalb sind sie 0 !

$\mathrm{div}\, \vec{P} = -\rho_P$ (über Vol der Materie einschl. $\int$)

$$\int_V d^3r\, \mathrm{div}\, \vec{P} = \underbrace{\oint_{\partial V} d^2\vec{f}\, \vec{P}}_{0 \text{ da } \vec{P} \text{ am Rand } = 0} = -\int_V \rho_P\, d^3 r$$

$\rightarrow \int_V \rho_P\, d^3 r = 0$

$\mathrm{rot}\, \vec{M} = \frac{1}{c} \vec{j}_M$

$$\int_F d^2\vec{f}\, \mathrm{rot}\, \vec{M} = \frac{1}{c}\int_F d^2\vec{f}\, \vec{j}_M$$

$$\underbrace{\int_{\partial F} d\vec{r} \cdot \vec{M}}_{0 \text{ da } \vec{M} \text{ am Rand } = 0} = \frac{1}{c}\int \vec{j}_M\, d^2\vec{f}$$

$\rightarrow \int \vec{j}_M\, d^2\vec{f} = 0$

D.h. pol. Gesamtlad $= 0$, kein $\vec{j}_M$ d.d. Schnittfl.

2. Schritt: $\vec{P}$ ist elektr. & $\vec{M}$ magn. Dipolmoment pro Vol.-Einheit

$$\operatorname{div}\vec{P} = \rho_P \qquad \int \underbrace{r_i \partial_j P_j}_{\partial_j(r_i P_j) - \underbrace{P_j \partial_j r_i}_{P_i}} = \int r_i \rho_P$$

$$\vec{r}\operatorname{div}\vec{P} = \vec{r}\rho_P$$

Term 0 am Rand

$$\rightarrow \int d^3r\, P(\vec{r}) = \int \vec{r}\rho_P(\vec{r})\, d^3r$$

$$\operatorname{rot}\vec{M} = \frac{1}{c}\vec{j}_M$$

$$\frac{1}{2}\int d^3r\, \underbrace{\vec{r}\times\operatorname{rot}\vec{M}} = \frac{1}{2c}\int d^3r\, \vec{r}\times\vec{j}_M$$

$$\varepsilon_{ijk} r_j \varepsilon_{klm}\partial_l M_m$$

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \text{ —"— } = r_m \partial_i M_m - r_l \partial_l M_i$$

1) $\partial_i(r_m M_m) - M_m \partial_i r_m$

2) $\partial_l(r_l M_i) - M_i \partial_l r_l$

$$\rightarrow \quad -M_i + 3M_i = 2M_i \checkmark$$

$$\rightarrow \int d^3r\, M_i = \frac{1}{2c}\int d^3r\, \vec{r}\times\vec{j}_M$$

Legt $\vec{P}$ & $\vec{M}$ erst eind. fest, sonst wäre es uneindeutig

<u>Materialgleichungen</u> historisch $\vec{P} = \vec{P}(\vec{E},\vec{H})$ statt $\vec{P}(\vec{E},\vec{B})$

Einfachste Näherung: $\vec{P} = \chi_e\vec{E}$ $\quad \vec{M} = \chi_m\vec{H}$

lineare N. — f. schwache & schwach veränd. Felder

$$\vec{E} = \vec{D} - 4\pi\vec{P}$$
$$\vec{B} = \vec{H} + 4\pi\vec{M}$$

$$\rightarrow \vec{D} = \vec{E} + 4\pi\vec{P}$$
$$\vec{D} = \vec{E} + 4\pi\chi\vec{E} = \vec{E}(1 + 4\pi\chi_e) = \varepsilon\vec{E}$$

$$\vec{B} = \vec{H}(1 + 4\pi \chi_m) = \mu \vec{H}$$

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{H} = \frac{1}{\mu}\vec{B}$$ // $\vec{B}$ & $\vec{E}$ sind primäre Feldgrößen

$$\vec{E} = \frac{\vec{D}}{\varepsilon}$$

$\varepsilon > 1$ dielektrisch

$\varepsilon \gg 1$ ferroelektrisch

schwächen d. $\vec{E}$ Feld im Inneren

$\vec{B} = \mu \vec{H}$ schwächt $\{ \mu < 1$ diamagnet zB 0,5

verstärkt $\begin{cases} \mu > 1 & \text{paramagnet zB 2} \\ \mu \gg 1 & \text{ferromagnet zB 100} \end{cases}$

χ: Suszeptibilität ε: Dielektrizitätsk.

μ: Permeabilitätsk.

ferro & ferro: $\mu(B)$ & $\varepsilon(E)$ außerdem nicht eindeutig

– Remanenz – Hysterese

Stromleitung NL: $\vec{j} = \emptyset$ bei $\vec{E}$ // da $\sigma = \emptyset$

Leiter: $\vec{j} = \sigma \vec{E}$

spez. Leitfähigkeit

RB an Grenzflächen 2er Medien

inh. $\partial_i D_i = 4\pi \rho$ hom $\partial_i B_i = \emptyset$

$$\text{rot } \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{D} \qquad \text{rot } \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$

$\rightarrow$ ✓ $\oint_{\partial V} d^2 f_i D_i = \int 4\pi \rho \, d^3 r$ | $\oint_{\partial V} d^2 f_i B_i = \emptyset$ ✓

✓ $\oint_{\partial F} dr_i H_i = \int_F \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{D} \, d^2\vec{f}$ | $\oint_{\partial F} d\vec{r}\,\vec{E} = -\int_F \frac{1}{c}\partial_t \vec{B} \, d\vec{f}$

$\rightarrow \quad 1 \rightarrow 2$ Medium

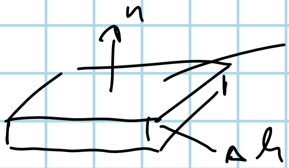
$$\vec n\,(\vec D_2 - \vec D_1) = 4\pi\,\sigma \qquad \vec n\,(\vec B_2 - \vec B_1) = 0$$

$$\vec n \times (\vec H_2 - \vec H_1) = \frac{4\pi}{c}\,\vec k \qquad \vec n \times (\vec E_2 - \vec E_1) = 0$$

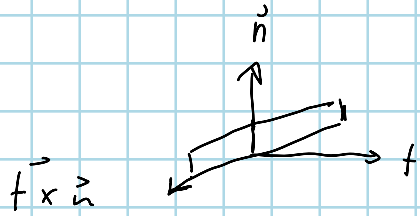
Beweis

$$d\vec f = \vec n\,\Delta a$$

$$\vec n(\vec D_2 - \vec D_1)\,\cancel{\Delta a} = 4\pi \underbrace{\int_V \rho\, dh}_{\sigma}\,\cancel{\Delta a}$$



$\Delta h \rightarrow 0 \qquad \Delta a \rightarrow 0$



$$\int_{\partial F} dr_i\, H_i = \int_F \left(\frac{4\pi}{c}\,\vec j + \frac{1}{c}\partial_t \vec D\right) d^2 f \qquad d\vec r = \vec t \times \vec n$$

$$(\vec t \times \vec n)(\vec H_2 - \vec H_1) = \frac{4\pi}{c}\underbrace{\int_F \vec j\, dh}_{\vec k} + \frac{1}{c}\partial_t(\vec D_2 + \vec D_1)\,\Delta h \qquad \checkmark$$

σ & $\vec k$ beides freie

Flächenlad. / strom-dichten

wenn σ & $\vec k = 0 \rightarrow \vec D_n$ stetig $\qquad \vec H_t$ stetig

$\vec B_n$ stetig $\qquad \vec E_t$ stetig

falls $\vec D$ & $\vec H$ bek. $\rightarrow \sigma$ & $\vec k$ bestimmen $\qquad \vec E_n$ & $\vec B_t$

i. A. unstetig

$$\partial_i P_i = -\rho_P \qquad \operatorname{rot} \vec M = \frac{1}{c}\,\vec j_M$$

$$\rightarrow \vec n\,(\vec P_2 - \vec P_1) = -\sigma_P \qquad \vec n \times (\vec M_2 - \vec M_1) = \frac{1}{c}\,\vec k_M$$

Grenzen der Anwendbarkeit

f. ε:

Mittelung muss viele Atome umf. $\sim 100\,Å$

$\lambda \gg a$ — Abst. d. Atome (räumlich)

$\rightarrow$ Frequ. kleiner als die langw. Röntgenstrahlung innerh. d. Mittelungsabst. dürfen sich freie Felder nicht stark ändern!

μ verliert früher d. Sinn — Momente brauchen Zeit sich auszubilden (endlich)

$\to$ Felder dürfen sich nicht zu schnell ändern

$$T = \frac{\lambda}{c} \gg T_{\text{umlauf}} \approx \frac{a}{v} \qquad m v^2 = \frac{e^2}{a}$$

$$\lambda \gg c\,T_{\text{uml}} = c\,\frac{a}{v} = c\,a\sqrt{\frac{ma}{e^2}} = a\sqrt{\frac{a}{r_{el}}}$$

$$r_{el} = \frac{e^2}{mc^2} = 3 \cdot 10^{-15}\,\text{m}$$

$a = 3\,\text{Å} \quad \to \lambda \gg 1000\,\text{Å}$

$\to \vec{D} = \mu H$ & $\vec{M} = \chi_m \vec{H}$ nur sinnvoll f. $f <$ sichtb. Licht!

optisch: $\mu = 1 \quad \vec{B} = \vec{H} \quad \vec{M} = \emptyset$ nur $\varepsilon \neq 1$

Beziehung zu Clausius Mosotti

$$\langle \vec{p}_{mol} \rangle = \chi_{mol} \langle \vec{E}_{frei} \rangle \qquad \text{// mikroskopisch}$$

Bez. zwischen χ_{mol} mikrosk. & ε makroskopisch

vom mol. gemittelte Feld (innerhalb) $\langle \vec{E}_{mol} \rangle = -\frac{1}{R^3} \int_{r'<R} d^3r'\, \rho_{mol}(\vec{r}')\, \vec{r}'$

$$= -\frac{4\pi}{3} \underbrace{\frac{1}{V} \int \text{—''—}}_{\vec{P}}$$

$$= -\frac{4\pi}{3} N \langle \vec{p}_{mol} \rangle$$

$$\vec{P} = N \langle \vec{p}_{mol} \rangle = N \chi_{mol} \langle \vec{E}_{frei} \rangle = N \chi_{mol} (\vec{E} - \langle \vec{E}_{mol} \rangle)$$

$$= N \chi_{mol} \left(\vec{E} + \frac{4\pi}{3} \vec{P} \right)$$

$$\to \vec{P} \left(1 - \frac{4\pi}{3} N \chi_{mol} \right) = N \chi_{mol} \vec{E}$$

$$\vec{P} = \underbrace{\frac{N \chi_{mol}}{1 - \frac{4\pi}{3} N \chi_{mol}}}_{\chi_e} \vec{E}$$

$$\Rightarrow \quad \varepsilon = 1 + 4\pi \chi_e = \frac{\overbrace{1 - \frac{4\pi}{3} N\chi + 4\pi N\chi}^{+2\frac{4\pi}{3}N\chi}}{1 - \frac{4\pi}{3} N\chi}$$

$\to$ χ_{mol} bestimmen mit ε (auf χ_{mol} umf.)

$$\varepsilon - \varepsilon \frac{4\pi}{3} N\chi = 1 + 2\frac{4\pi}{3} N\chi$$

$$\varepsilon - 1 = \varepsilon \frac{4\pi}{3} N\chi + 2 \frac{4\pi}{3} N\chi$$

$$= \chi (\varepsilon + 2) \frac{4\pi}{3} N$$

$$\to \chi = \frac{\varepsilon - 1}{(\varepsilon + 2) \frac{4\pi}{3} N} \quad \checkmark$$

nur f. kleine ε-Werte ok

Orientierungspol.

$$\vec{P} = N \langle \vec{p}_{mol} \rangle$$

$\langle \vec{p}_{mol} \rangle = \chi_{mol} \langle \vec{E}_{frei} \rangle$

hier $\vec{E}$ wirkt in $\vec{e}_z$ Richtung

$$= N p \underbrace{\langle \cos\vartheta \rangle}_{\text{therm. MW}} \vec{e}_z$$

$$W^{ww}(\vartheta) = -p E_{wirk} \cos\vartheta \qquad // -\vec{p}\vec{E}$$

$$\langle \cos\vartheta \rangle = \frac{\int d\Omega \cos\vartheta \, e^{-\frac{W^{ww}}{kT}}}{\int d\Omega \, e^{-\frac{W^{ww}}{kT}}} \qquad // \text{Boltzmann}$$

$$e^{-\frac{W^{ww}}{kT}} = 1 - \frac{W^{ww}}{kT}$$

$$= \frac{\int d\Omega \cos\vartheta - \int d\Omega \frac{W}{kT}\cos\vartheta}{\int d\Omega\, 1 - \frac{W}{kT}}$$

$\int d\Omega \cos\vartheta = 0$

$\int d\Omega \cos^2\vartheta =$

$\frac{1}{3}\int d\Omega$

$$= \frac{\frac{1}{kT}\int d\Omega\; pE\cos^2\vartheta}{4\pi + \underbrace{\int d\Omega\, pE\cos\vartheta}_{\emptyset}}$$

$$= \frac{\frac{1}{3kT}\int d\Omega\; pE}{4\pi} = \frac{pE}{3kT}$$

$$\vec{P} = N\, p \langle\cos\vartheta\rangle\, \vec{e}_z = N\, p^2 \frac{E_{\text{wirk}}}{3kT}\vec{e}_z$$

$$\chi_e \propto \frac{1}{T}$$

$\rightarrow$ Ausrichtung fällt mit Temp!

Kramers-Kronig Rel.

$$\vec{D} = \vec{E} + 4\pi\vec{P} \quad \text{— alles } (\vec{r},t)$$

$$4\pi\vec{P}(\vec{r},t) = \int_0^\infty d\tau\; G(\tau)\,\vec{E}(\vec{r},t-\tau)$$

Fouriertrafo $\int dt\, e^{i\omega t}$

$$\vec{P}(\vec{r},\omega) = \frac{1}{4\pi}\int_0^\infty d\tau\; G(\tau)\, e^{i\omega\tau}\underbrace{\int \overset{dt'}{\overset{\shortparallel}{dt}}\; e^{i\omega\overbrace{(t-\tau)}^{t'}}\vec{E}(\vec{r},\overbrace{t-\tau}^{t'})}_{E(\vec{r},\omega)}$$

$$\vec{P}(\vec{r},\omega) = \chi_e\,\vec{E}(\vec{r},\omega)$$

$$\chi_e = \frac{1}{4\pi}\int_0^\infty d\tau\; G(\tau)\, e^{i\omega\tau}$$

$G(\tau)$ einfl. v $\vec{E}$ zu früheren Zeiten — wird < mit $>\tau$

$$\varepsilon = 1 + 4\pi\chi_e = 1 + \int \ldots \quad \checkmark$$

Symmetrie: $G(T)$ ~~ist~~ reell weil $\vec{E}$ reell

$$\varepsilon^* = \varepsilon(-\omega)$$ // nur $e^{\sim}$ komplex

$$\varepsilon(\omega) = \mathrm{Re}(\varepsilon) + i\,\mathrm{Im}(\varepsilon)$$

$$\varepsilon^*(\omega) = \mathrm{Re}(\varepsilon) - i\,\mathrm{Im}(\varepsilon) = \varepsilon(-\omega) = \mathrm{Re}\,\varepsilon(-\omega) + i\,\mathrm{Im}\,\varepsilon(-\omega)$$

$$\rightarrow \mathrm{Re}\,\varepsilon(\omega) = \mathrm{Re}\,\varepsilon(-\omega) \qquad \mathrm{Im}\,\varepsilon(-\omega) = -\mathrm{Im}\,\varepsilon(\omega) \quad \checkmark$$

analytisch

$$\varepsilon(\omega_r + i\omega_i) = 1 + \int_0^\infty dT\, G(T)\, \underbrace{e^{i(\omega_r + i\omega_i)T}}_{e^{i\omega_r T} e^{-\omega_i T}}$$

$\int \exists$ f. ω_i pos $\rightarrow$ obere Halbebene

Cauchy: $$f(z) = \frac{1}{2\pi i} \oint_C d\omega \frac{f(\omega)}{\omega - z}$$

$$\rightarrow \varepsilon(z) - 1 = \frac{1}{2\pi i} \oint d\omega \frac{\varepsilon(\omega) - 1}{\omega - z}$$

$\varepsilon(\omega)$ wird 1 f $|\omega| \rightarrow \infty$ $\rightarrow$ C auf reelle Achse drücken

Trick $$\int_{-\infty + i\eta}^{\infty + i\eta} \rightarrow \int_{-\infty}^{\infty}$$ $i\eta$ in ω einbauen?

$$\omega = \omega' - i\eta$$

$$\varepsilon(\omega) = 1 + \lim_{\eta \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon(\omega') - 1}{\omega' - \omega - i\eta}$$

$$\lim_{\eta \to 0} \frac{1}{\omega' - \omega - i\eta} = P \frac{1}{\omega' - \omega} + i\pi\,\delta(\omega - \omega')$$

$$\rightarrow \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} P \frac{1}{\omega'-\omega}(\varepsilon(\omega')-1) + \underbrace{\int_{-\infty}^{\infty} i\pi\,\delta(\omega-\omega')(\varepsilon(\omega')-1)\,d\omega'}_{i\pi\,(\varepsilon(\omega)-1)}\right.$$

$$\varepsilon(\omega) = 1 + \frac{1}{2\pi i} \int P \frac{\varepsilon(\omega')-1}{\omega'-\omega} + \frac{1}{2\cancel{\pi i}}\cancel{i\pi}(\varepsilon(\omega)-1)$$

$$\varepsilon(\omega) = 1 + \frac{1}{2\pi i}\int - " - + \frac{1}{2}\varepsilon(\omega) - \frac{1}{2}$$

$$\frac{1}{2}\varepsilon(\omega) = \frac{1}{2} - \frac{i}{2\pi}\int - " -$$

$$\rightarrow \varepsilon(\omega) = 1 - \frac{i}{\pi}\int " \ldots \quad \checkmark$$

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + \frac{1}{\pi} P \int \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'-\omega}$$

analog f. $\mathrm{Im}\,\varepsilon(\omega)$

Beschr auf pos ω

$$\mathrm{Re}(\omega) = \mathrm{Re}(-\omega) \qquad \mathrm{Im}(-\omega) = -\mathrm{Im}(\omega)$$

$$\rightarrow \mathrm{Re}\,\varepsilon(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{0} \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'-\omega} + \int_{0}^{\infty} \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'-\omega}$$

$$\omega' = -\omega'$$

$$\rightarrow -\int_{\infty}^{0} \frac{\mathrm{Im}\,\varepsilon(-\omega')}{-\omega'-\omega}\,d\omega' \rightarrow \int_{0}^{\infty} \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'+\omega} \quad T1$$

$$\begin{matrix} T1 \\ +T2 \end{matrix} \longrightarrow \int_{0}^{\infty} \frac{\mathrm{Im}\,\varepsilon(\omega'+\omega) + \mathrm{Im}\,\varepsilon(\omega'-\omega)}{\omega'^2-\omega^2} = \int_{0}^{\infty} \frac{\mathrm{Im}\,\varepsilon\, 2\omega'}{\omega'^2-\omega^2}$$

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + \frac{2}{\pi} P \int_{0}^{\infty} \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'^2-\omega^2} \quad \checkmark$$

Leiter $\lim_{T\to\infty} G(T) = 4\pi\sigma$

η : komplex

$$\mathrm{Re}\left(\eta - i\frac{4\pi\sigma}{\omega}\right) = \mathrm{Re}\,\eta \qquad \mathrm{Im}\left(\eta - \frac{i4\pi\sigma}{\omega}\right) = \mathrm{Im}\,\eta - \frac{4\pi\sigma}{\omega}$$

Jetzt alles einsetzen → fertig

Normale / anormale Dispersion

$\operatorname{Im}\varepsilon(\omega) = 0$ (f. einen Transparenzbereich)

$$\operatorname{Re}\varepsilon(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\operatorname{Im}\varepsilon(\omega')}{\omega'^2 - \omega^2}$$

$$\frac{d}{d\omega} \operatorname{Re}\varepsilon(\omega) = \frac{2}{\pi} P \int_0^\infty \omega' \operatorname{Im}\varepsilon(\omega') \frac{d}{d\omega} \frac{1}{\omega'^2 - \omega^2}$$

// P & $\frac{d}{d\omega}$ vertauschbar, da in Nähe v ω kein Beitrag → $\int$

$$\frac{d}{d\omega}(\,-\,)^{-1} = -1(\quad)^{-2} \cdot -2\omega$$

$$\rightarrow \frac{d}{d\omega} \operatorname{Re}\varepsilon(\omega) = \frac{4\omega}{\pi} P \int_0^\infty \frac{\omega' \operatorname{Im}\varepsilon(\omega')}{(\omega'^2 - \omega^2)^2} > 0$$

→ normale Disp.

$\frac{d}{d\omega} \operatorname{Im}\varepsilon(\omega) < 0$ anormale Disp.

$\operatorname{Im} \ll \operatorname{Re}$ Transparenzbereiche

Oszillatormodell:

gebe e als Oszill.

$$m(\ddot{\vec{r}} - \gamma\dot{\vec{r}} + \omega_0^2 \vec{r}) = \vec{F} = e\vec{E}$$

$\vec{p}$: Dipolmoment statt $\vec{r}$ $\vec{p}$ $p = p(\omega)$

$$m(\omega^2 - i\omega\gamma + \omega_0^2)\,\vec{p}(\omega) = e^2 \vec{E}$$

$$\rightarrow \vec{p} = \frac{e^2}{m(-\omega^2 - i\omega\gamma + \omega_0^2)} \vec{E}$$

analytisch oben
$\operatorname{Im}\omega > 0$

$$\vec{P} = N\vec{p} = \underbrace{\frac{Ne^2}{m} \frac{1}{-\omega^2 - i\gamma\omega + \omega_0^2}}_{\chi_e(\omega)} \vec{E} \longrightarrow \varepsilon(\omega) = 1 + 4\pi\chi_e$$

Energie & Impulsbilanz

mikroskopisch

$$\left.\begin{aligned} f^\mu &= \frac{1}{c} F^{\mu\nu} j_\nu^{tot} \\ \partial^\mu F_{\mu\nu} &= \frac{4\pi}{c} j_\nu^{tot} \end{aligned}\right\} \longrightarrow \frac{1}{c} f^\mu = -\partial_\sigma T^{\sigma\mu}$$

$$T^{\sigma\mu} = \frac{1}{4\pi c}\left(F^{\sigma\eta} F_\eta{}^\mu - \frac{1}{4} g^{\sigma\mu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

jetzt interessant: makroskopische EB

$$\underbrace{f^\mu_{(mat)} = \frac{1}{c} F^{\mu\nu} j_\nu}_{\text{freie Quellen}} \quad \& \quad \partial^\mu H_{\mu\nu} = \frac{4\pi}{c} j_\nu$$

$$f^\mu_{(mat)} = \frac{1}{c} F^{\mu\nu} \frac{c}{4\pi} \partial^\sigma H_{\sigma\nu} \Rightarrow \frac{1}{c} f^\mu_{(mat)} =$$

$$\frac{1}{4\pi c} \partial^\sigma (F^{\mu\nu} H_{\sigma\nu}) - \frac{1}{4\pi c} \underbrace{H_{\sigma\nu} \partial^\sigma F^{\mu\nu}}_{NR}$$

$$H_{\sigma\nu} \partial^\sigma F^{\mu\nu} = H_{\nu\sigma} \partial^\nu F^{\mu\sigma} = H_{\sigma\nu} \partial^\nu F^{\sigma\mu} = \frac{1}{2} H_{\sigma\nu} \underbrace{(\partial^\sigma F^{\mu\nu} + \partial^\nu F^{\sigma\mu})}_{-\partial^\mu F^{\nu\sigma}}$$

$$\Rightarrow f^\mu_{(mat)} = \frac{1}{4\pi c}\left[\partial^\sigma (F^{\mu\nu} H_{\sigma\nu}) + \frac{1}{2} H_{\sigma\nu} \partial^\mu F^{\nu\sigma}\right]$$

$$= -\frac{1}{4\pi c}\left[\partial_\sigma (F^{\mu\nu} H_\nu{}^\sigma) + \frac{1}{2} H_{\sigma\nu} \partial^\mu F^{\sigma\nu}\right]$$

$$\left.\begin{aligned} \partial^\sigma &= g^{\sigma\eta} \partial_\eta \\ g^{\sigma\eta} H_{\sigma\nu} &= H^\eta{}_\nu \end{aligned}\right\}$$

$\neq$ lok. Bilanzgl. / ∂_σ nicht herausziehbar!

Minkowski Tensor / konstante ε, μ

$H^{\mu\nu} \approx \vec{D}, \vec{H}$

$F^{\mu\nu} \approx \vec{E}, \vec{B}$

$\vec{B} = \mu \vec{H} \qquad \vec{E} = \frac{\vec{D}}{\varepsilon}$

$H^{0i} = \varepsilon F^{0i} \qquad H^{ij} = \frac{1}{\mu} F^{ij}$

$$f^\mu_{(mat)} = -\frac{1}{4\pi c}\left[\partial_\sigma\left(F^{\mu\nu}H_\nu{}^\sigma\right) + \underbrace{\frac{1}{2} H_{\sigma\nu}\partial^\mu F^{\sigma\nu}}_{\frac{1}{4}\partial^\mu\left(H_{\rho\nu}F^{\rho\nu}\right)}\right]$$

$$\to g^{\mu\sigma}\partial_\sigma$$

$$= -\frac{1}{4\pi c}\,\partial_\sigma\left[\left(H^{\sigma\nu}F_\nu{}^\mu\right) + \frac{1}{4} g^{\mu\sigma}\left(H_{\rho\nu}F^{\rho\nu}\right)\right]$$

$$= -\frac{1}{4\pi c}\,\partial_\sigma T^{\mu\sigma}_{(mat)}$$

nur lok. B G, da $T_{(mat)}$ nicht symmetrisch

$$T^{\mu\sigma}_{(mat)} = \begin{pmatrix} \frac{1}{c} w_{(mat)} & \vec{g}_{(mat)} \\ \frac{1}{c^2}\vec{S}_{(mat)} & -\frac{1}{c} T^{ij}_{(mat)} \end{pmatrix}$$

T muss nur bei abgeschl. System symmetr. sein!

$$w_{mat} = \frac{1}{8\pi}\left(\vec{E}\vec{D} + \vec{H}\vec{B}\right)$$

$$F^{0i} = -E^i \qquad H^{0i} = -D^i$$

$$F^{ij} = -\varepsilon_{ijk}B_k \qquad H^{ij} = -\varepsilon_{ijk}H_k$$

$$\vec{S}_{mat} = \frac{c}{4\pi}\left(\vec{E}\times\vec{H}\right)$$

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$T^{j0}_{(mat)} = H^{ji}F_i{}^0 + \underbrace{\frac{1}{4} g^{j0} H_{\rho\nu}F^{\rho\nu}}_{\text{fällt weg}}$$

$$= +E^i\varepsilon_{ijk}H_k$$

$$g = \frac{1}{4\pi c}\left(\vec{D}\times\vec{B}\right) \quad \checkmark$$

$$T^{0j}_{(mat)} = H^{0i}F_i{}^j$$

<u>Dissipative Medien</u>